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FINITE p-GROUPS NOT CHARACTERISTIC IN ANY p-GROUP IN WHICH THEY ARE PROPERLY CONTAINED

BY

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ABSTRACT

Answering a question raised by Y. Berkovich, we give examples of finite p-groups G with the property that the only finite p-group K with $G\,char\,K$, is G itself. We also prove a theorem stating that every finite p-group is contained in such a group G.

Let p be a prime. Y. Berkovich has recently raised the question whether there is a finite p-group G which is not characteristic in any finite p-group properly containing it. Since the world of finite p-groups is densely populated, theorems governing all of its inhabitants are rare, and it seems likely that questions like this are raised in the hopes of a negative answer. However, we are going to prove

Theorem: *For every finite p-group G there is a finite p-group H such that $G \leq H$ and H is not characteristic in any finite p-group properly containing it.*

This theorem suggests that the answer to the question what makes a finite p-group G characteristic in another finite p-group, K, partially lies within G itself. It has been known for a long period (see [2]) that a finite p-group G

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has outer automorphisms of p-power order; the theorem indicates the possibility that, given any finite p-group G, one cannot form arbitrarily long chains $G = G_1 \, char \, G_2 \, char \, \ldots \, char \, G_n$ of extensions by outer p-automorphisms.

We are first going to present examples of finite p-groups that are not characteristic in any finite p-group in which they are properly contained. These examples I believe to be as small as possible; this will be commented on after the reader has seen the examples. The theorem as such will be proved inductively, using a wreath product construction. The cases $p = 2$ and p odd must be handled separately throughout.

NOTATION: The i-th term of the lower central series of the group P will be denoted by $\gamma_i(P)$. For group elements x and y, we let $[x, y] = [x, {}_1y]$, $[x, {}_ny] = [[x, {}_{n-1}y], y]$, $2 \le n \in \mathbb{N}$. We will use "$<\cdot$" to say "is a maximal subgroup of". We will otherwise be using the notation introduced in Chapter 5 of [3].

We will be using the following corollary to the Hall–Petrescu formula, to be found e.g. in [1] (11.9): If x, y are elements of the group H, then

$$(*) \qquad (xy)^{p^j} \equiv x^{p^j} y^{p^j} \bmod \mho_j(\gamma_2(H)) \prod_{\ell=1}^{j} \mho_{j-\ell}(\gamma_{p^\ell}(H))$$

Remark 1: Some facts on automorphisms of semidirect products and wreath products.

a) First of all, let $H = QS$ and $\widehat{H} = \widehat{Q}\widehat{S}$ be products of the normal subgroups Q, $\widehat{Q}$ and the subgroups S and $\widehat{S}$, respectively. If $\sigma_1 : Q \to \widehat{Q}$ and $\sigma_2 : S \to \widehat{S}$ are isomorphisms satisfying the conditions

$\sigma_1 |_{Q \cap S}$ and $\sigma_2 |_{Q\cap S}$ both induce the same isomorphism $Q \cap S \to \widehat{Q} \cap \widehat{S}$ and

$$(1) \qquad (q^{\sigma_1})^{s^{\sigma_2}} = (q^s)^{\sigma_1}, \quad \text{whenever } s \in S \text{ and } q \in Q$$

then there is an isomorphism $\sigma : H \to \widehat{H}$ given by $(qs)^\sigma = q^{\sigma_1} s^{\sigma_2}$. Indeed, the first condition in (1) makes σ well-defined, while the second yields $(sq)^\sigma (tr)^\sigma = s^{\sigma_2} q^{\sigma_1} t^{\sigma_2} r^{\sigma_1} = (st)^{\sigma_2} (q^{\sigma_1})^{t^{\sigma_2}} r^{\sigma_1} = (st)^{\sigma_2} (q^t r)^{\sigma_1} = (stqr)^\sigma$, whenever q, $r \in Q$ and s, $t \in S$.

b) Let $H = R \wr S$ be the wreath product of the groups R and S with respect to a faithful transitive permutation representation of S on some set Ω, and let $Q = R^\Omega$ be its base group. Let us identify S with its image in S_Ω. If $\tau \in N_{S_\Omega}(S)$ and $f \in Q$, define $f^{\tau'}$ via setting $f^{\tau'}(\omega) = f(\omega^{\tau^{-1}})$, $\omega \in \Omega$. For $t \in S$, $f \in Q$,

$\omega \in \Omega$, we obtain $(f^{\tau'})^{t^\tau}(\omega) = f^{\tau'}(\omega^{t^{-\tau}}) = f(\omega^{\tau^{-1}t^{-1}}) = f^t(\omega^{\tau^{-1}}) = f^{t\tau'}(\omega)$. According to a), this entails that H has an automorphism defined by $(sf)^\tau = s^\tau f^{\tau'}$, $s \in S$, $f \in Q$.

c) Let H, Q be as in b) and suppose that $Q\, char\, H$. For $\omega \in \Omega$ let R_ω be the set $\{f \in R^\Omega : f(\nu) = 1 \text{ whenever } \omega \neq \nu \in \Omega\}$. If $f \in Q$, then $C_Q(f)$ is isomorphic to the direct product of the groups $C_{R_\omega}(f(\omega))$, $\omega \in \Omega$. Thus if $\tau \in \mathrm{Aut}(H)$, $\omega \in \Omega$ and $f \in R_\omega$ then there is $\nu \in \Omega$ with $f^\tau \in R_\nu Z(Q)$. Letting $f, g \in R_\omega \setminus Z(R_\omega)$ and applying this argument to f, g, fg, we see that there is $\nu \in \Omega$ with f^τ, $g^\tau \in R_\nu Z(Q)$. In other words, $\langle\tau\rangle$ acts on the set $\{R_\omega Z(Q) : \omega \in \Omega\}$.

d) Now let p be a prime, and R a finite p-group, let $S = \langle\alpha\rangle \cong \mathbb{Z}/p^n\mathbb{Z}$ and let H be the regular wreath product of R and S. Let $\langle\beta\rangle$ be a complement of Q in H, let $Q_1 = Q$, $Q_{i+1} = \Phi(Q_i)$, $i \in \mathbb{N}$, and suppose that there is $x \in Q$ with $\langle\beta\rangle^x = \langle\alpha y\rangle$, $y \in Q_i$. From $o(\alpha y) = p^n$ and $(*)$, we derive $[y, {}_{p^n-1}\alpha] \in Q_{i+1}$. Regarded as an $\langle\alpha\rangle$-module, $V := Q_i/Q_{i+1}$ is a direct sum of isomorphic copies of $GF(p)[\langle\alpha\rangle]$. This entails that $yQ_{i+1} \in [V, \alpha]$; letting $yQ_{i+1} = [\alpha, v]$, $v = uQ_{i+1}$, we obtain that $\beta^{xu^{-1}} \in \langle\alpha\rangle Q_{i+1}$. Via induction on i, we find that $\langle\alpha\rangle$ and $\langle\beta\rangle$ are actually Q-conjugates, in particular $\mathrm{Aut}(H) = \mathrm{Inn}(H)N_{\mathrm{Aut}(H)}(\langle\alpha\rangle)$.

Lemma 2: *Let p be an odd prime and let G be the semidirect product of $\langle\alpha\rangle$ with $\langle\iota\rangle$, where $o(\alpha) = p^3$, $o(\iota) = p^2$, $\alpha^\iota = \alpha^{p+1}$. The group G is not characteristic in any finite p-group properly containing it.*

Proof. Observe that $\gamma_2(G) = \langle\alpha^p\rangle$, $\gamma_3(G) = \langle\alpha^{p^2}\rangle = Z(G)$, $\Phi(G) = \langle\alpha^p\rangle\langle\iota^p\rangle$.

Using $(*)$, we obtain that

(2) For $r, s, t \in \mathbb{Z}, (\alpha^r\iota^s)^p \equiv \alpha^{rp}\iota^{sp}(\bmod \langle\alpha^{p^2}\rangle)$,

while $[\alpha\iota^{tp}, \iota] = \alpha^p = (\alpha\iota^{tp})^p$.

Using (2), we find that $\Omega_2(G) = \langle\iota, \alpha^p\rangle$, while the elements g of G enjoying the property that $o(g) = p^3$ and there is x in G satisfying $g^x = g^{p+1}$ are precisely the elements $\alpha^r\iota^{sp}$ with r not divisible by p.

Let $H \in Syl_p(\mathrm{Aut}(G))$; both $\langle\iota\rangle\Phi(G)$ and $\langle\alpha\rangle\Phi(G)$ having just been seen to be characteristic in G, we have $[G, H] \leq \Phi(G)$. Let $\vartheta \in H \setminus N_H(\langle\alpha\rangle)$; upon maybe replacing ι be some p'-power, there is $k \in \mathbb{N}$ such that $\alpha^\vartheta = \alpha^{kp+1}\iota^p$, whence $\alpha^{\vartheta\tilde{\iota}} = \alpha\iota^p$ for some power $\tilde{\iota}$ of ι and we may assume $\alpha^\vartheta = \alpha\iota^p$. By

(2), this entails $[\alpha^p, \vartheta] = 1$, whence $[\alpha\iota^p, \iota^\vartheta] = \alpha^p$. Thus $\iota^\vartheta = \iota\alpha^{p\ell}$ for some ℓ. Replacing ϑ by the automorphism $g \mapsto g^{\alpha^i\vartheta}$ for a suitable i gets us to $\iota^\vartheta = \iota$, $\alpha^\vartheta = \alpha\iota^p$. Note that $[\alpha, \vartheta^p] = [\alpha, \vartheta]^p = 1$, whence $\vartheta^p = \mathrm{id}$.

If $\beta \in N_H(\langle\alpha\rangle)$, then $[\alpha, \beta\iota^k] = 1$ for some k, as, indeed, $\langle\iota\rangle$ induces the Sylow-p-subgroup of $\mathrm{Aut}(\langle\alpha\rangle)$ on $\langle\alpha\rangle$. We may thus presume $\beta \in C_H(\alpha)$. Then $[\beta, \iota] \in C_{\Omega_2(G)}(\alpha) = \langle\alpha^p\rangle$, which shows $\beta \in \mathrm{Inn}(G)$.

As $H = \langle\vartheta, N_H(\langle\alpha\rangle)\rangle$, $H = \langle\vartheta, \mathrm{Inn}(G)\rangle$. The group G must have non-inner p-automorphisms ([2]), so ϑ exists; we might also apply Remark 1a).

Let $G \triangleleft K$ be a finite p-group. We would like to produce an automorphism σ of K that does not normalise G.

Let $C = C_K(G)$. The description of $\mathrm{Aut}(G)$ just procured yields that $K = GC\langle t\rangle$, where either $t = 1$ or t induces ϑ on G. Hence $t^p \in C$, $[t, \langle\iota, \Phi(G)\rangle] = 1$, $C\langle t\rangle \cap G \leq \langle\alpha^{p^2}\rangle$. If $C \not\leq G$, then there is $x \in C$ such that $\langle x^p, [x, K]\rangle \leq Z(G) = \langle\alpha^{p^2}\rangle$. Let $\alpha^\sigma = \alpha x$. There is $\ell \in \{0, \dots, p-1\}$ satisfying $x^p = \alpha^{\ell p^2}$. Then $(\alpha x)^p = \alpha^p x^p = [\alpha x, \iota\iota^{\ell p}]$. This amounts to saying that the maps $\sigma_1 : \alpha \mapsto \alpha x$, $\sigma_2 : \iota \mapsto \iota\iota^{\ell p}$ satisfy (1), thus Remark 1a) says that the map $\alpha^i\iota^j \mapsto \alpha^{i\sigma_1}\iota^{j\sigma_2}$ is an isomorphism.

Let $U = \langle C, t, \iota^p\rangle = \langle C, t\rangle \times \langle\iota^p\rangle$. If $u \in U$, there is $\ell_u \in \{0, \dots, p-1\}$ satisfying $[y, u] = \alpha^{p^2\ell_u}$. It is well-known (and easily checked) that, if H is any finite p-group, if $z \in \Omega_1(Z(H))$ and $H_1 <\cdot\, H = \langle H_1, y\rangle$, there is an automorphism of H centralising H_1 and mapping y to yz. Note that $\langle\iota^p, \alpha^{p^2}\rangle = U \cap G \leq C_U(x)$ and $|U : C_U(x)| \leq p$. Hence there is $\sigma_2 \in \mathrm{Aut}(U)$ defined by $u^{\sigma_2} = u\iota^{-p\ell_u}$, $u \in U$. Furthermore, $[\alpha y, u^{\sigma_2}] = [\alpha, u]\alpha^{-p^2\ell_u}[y, u] = [\alpha, u]$, while $[\iota\iota^{\ell p}, u^{\sigma_2}] = [\iota, u] = 1$. Since $[G, \langle\vartheta\rangle] = \langle\iota^p, \alpha^{p^2}\rangle = U \cap G \leq C_G(\sigma_1) \cap C_U(\sigma_2)$, these considerations imply that

$$[u, g] = [u^{\sigma_2}, g^{\sigma_1}] = [u, g]^{\sigma_i} \in Z(U).$$

whenever $u \in U$ and $g \in G$, $i = 1, 2$. The maps σ_1 and σ_2 satisfy the requirements of (1), thus Remark 1a) says that σ, defined by $(ug)^\sigma = g^{\sigma_1}u^{\sigma_2}$, $u \in U$, $g \in G$, is in $\mathrm{Aut}(K)$.

It remains to investigate the possibility that $K = \langle G, t\rangle$, conjugation by t inducing ϑ on G. Let $t^p = \alpha^{p^2\ell}$, $\ell \in \{0, \dots, p-1\}$. If $p > 3$, then, as $[\alpha, t, t] = 1$ and $\langle[t, \alpha, \alpha]\rangle = \langle\alpha^{p^2}\rangle$ while $[\iota, t] = 1$ anyway, we have $\gamma_p(\langle G, t\rangle) = 1$, in particular $(\alpha t)^p = \alpha^p t^p = \alpha^p\alpha^{p^2\ell}$ by $(*)$. Thus $[\alpha t, \iota\iota^{\ell p}] = [\alpha, \iota]\alpha^{p^2\ell} = (\alpha t)^p$. Applying Remark 1a) yields an isomorphism σ mapping α to αt, ι to $\iota\iota^{\ell p}$. Since

$[\iota\iota^{\ell p}, t] = 1 = [\iota, t]$ and $(\alpha t)^t = \alpha t \iota^p = \alpha t(\iota\iota^{\ell p})^p$, we may set $t^\sigma = t$ and refer to Remark 1a) to extend σ into an automorphism of K.

If $p = 3$, then $(\alpha t)^3 = \alpha^3 t^3 [\iota^{-3}, \alpha] = \alpha^{3+9\ell} \cdot \alpha^9$, whence $[\alpha t, \iota\iota^{3(\ell+1)}] = [\alpha, \iota]\alpha^{9(\ell+1)} = (\alpha t)^3$. Define σ via $\alpha \mapsto \alpha t$, $\iota \mapsto \iota\iota^{3(\ell+1)}$, $t \mapsto t$. ∎

Remark 3: If p is odd and F is a p-group of order less than p^5, then F is characteristic in some finite p-group properly containing F.

A short justification: Let F be a finite p-group of order at most p^4; if $p > 3$ or if $cl(F) \leq 2$, then F is regular. Assume that $|\mho_1(F)| \leq p$. Then $\Omega_1(F)$ has index at most p in F and $(xy)^p = x^p$ whenever $y \in \Omega_1(F)$ and $x \in F$. Letting $1 \neq z \in \Omega_1(Z(F))$, $K = F\langle c \rangle$, $[F, c] = 1$, $c^p = z$, we see that $F \, char \, K$. If $p = 3$ and F is of maximal class, then $A = C_F(F')$ is abelian of order 3^3. Let $F = A\langle x \rangle$, and let $K = F\langle t \rangle$ with $\langle t^3, [t, \langle x \rangle F'] \rangle = 1$ and $[A, t] = Z(F)$; in K, A is no longer characteristic, so neither is F. Certainly, F cannot be abelian, so $|\mho_1(F)| = p^2$, and F is metacyclic. Let $F = \langle x \rangle\langle y \rangle$ with $\langle x \rangle \lhd F$, and either $o(x) = p^2$ and $x^y = x^{p+1}$ or $o(x) = p^3$ and $x^y = x^{p^2+1}$. In the second case, $F \, char \, G_1$ with $G_1 \cong G$, ($F \cong \langle \alpha, \iota^p \rangle$ being characteristic in G by (2)). In the first case, there is $\tau \in \mathrm{Aut}(F)$ with $x^\tau = xy$, $y^\tau = y$. If $|F| = p^4$, then let H be the semidirect product $F\langle \tau \rangle$. Then $\mho_1(H) = \mho_1(F)$, and $F = C_H(\mho_1(H)) \, char \, H$. If $|F| = p^3$ and $p > 3$, then let $H = F\langle t \rangle$ with $t^p = x^p$ and t inducing τ on F. For $0 \leq i, j \leq p-1$, $0 \leq k \leq p^2 - 1$, $(x^k y^i t^j)^p = x^p t^{jp} \neq x^p$ using $(*)$, and no automorphism of H could map x to $x^k y^i t^j$. If $|F| = 3^3$, then let $H = F\langle \tau \rangle$ be the semidirect product. Suppose that $x^\sigma = uv$, $u = x^i z$ $v = t^j$, $i, j \in \{1, -1\}$, $z \in \langle y, x^3 \rangle$; from $x^3 = x^{3i}[v, x^i, x^i]$, we derive $i = -1 = j$. If 3 does not divide i, then $C_H(x^i tz)$, $z \in \langle y, x^3 \rangle$, is equal to $\langle x^i tz, x^3 \rangle$, so $t^\sigma \in t\langle y, x^3 \rangle$. Thus $(xt)^\sigma \in x^{-1}\langle y, x^3 \rangle$, so $o((xt)^\sigma) = 9 \neq 3 = o(xt)$. Accordingly, $\langle x^H \rangle = F \, char \, H$.

Lemma 4: *Let $p = 2$, let $A = \langle a, b, c \rangle$ be homocyclic of exponent 4 and rank 3, let $G = \langle A, \alpha, \iota \rangle$ where*

$$a^\alpha = ab, \; b^\alpha = bc, \; c^\alpha = ca^2,$$
$$d^\iota = d^{-1}, \; d \in A,$$
$$\alpha^8 = 1 = \iota^4 = [\alpha, \iota], \; \iota^2 = c^2.$$

Then if K is a finite 2 group with $G < K$, G is not characteristic in K.

Proof. Let $\{a_1, b_1, c_1\}$ be any set of generators of A; then there is $\varphi \in \mathrm{Aut}(A)$ with $d^\varphi = d_1$, $d \in \{a, b, c\}$ — A being the free abelian group of rank 3 and exponent 4, after all. Right now this remark is directed at α, yet we will bear it in mind.

Note that $C_A(\alpha) = Z(G) = \langle c^2\rangle$, while $\Phi(G) = \langle \alpha^2, b, c, \Phi(A)\rangle$ and $G' = [A, G] = \langle b, c, \Phi(A)\rangle$. Furthermore, $[a, \alpha^4] = b^2c^2$ and $[b, \alpha^4] = c^2$, while $C_A(\alpha^4) = \langle c, \Phi(A)\rangle$ and $[b, \alpha^2] = a^2c^2$. Thus $o(\alpha) = 8$, $C_G(b) = A$, and $A = C_G(G')\, char\, G$. This implies that $\langle A, \iota, \alpha^4\rangle = C_G(A/\Phi(A))$ is likewise characteristic, whence $\mathrm{Aut}(G)$ normalises the chain $\Phi(G) <\cdot A\Phi(G) <\cdot A\Phi(G)\langle\iota\rangle$ and thus is a 2-group.

Let $d = a^ib^jc^k$ with $i, j, k \in \mathbb{Z}$ be any element of A. Then $[d, \alpha] = b^ic^ja^{2k}$, $[d, \alpha, \alpha] = c^ia^{2j}b^{2k}$, and $[d, \alpha, \alpha, \alpha] = a^{2i}b^{2j}c^{2k} = d^2$. Thus if α' is an $\mathrm{Aut}(A)$-conjugate of α, then $[a, \alpha', \alpha', \alpha'] = a^2$. If α' is contained in the unique Sylow-2-subgroup of $\mathrm{Aut}(A)$ stabilising the flag $\Phi(A) <\cdot \Phi(A)\langle c\rangle <\cdot \Phi(A)\langle c, b\rangle <\cdot A$ and $[a, \alpha', \alpha', \alpha'] = a^2$, then

$$(3) \quad \alpha^\psi = \alpha', \text{ with } \psi \in \mathrm{Aut}(A) \text{ defined by } a^\psi = a, b^\psi = [a, \alpha'], c^\psi = [a, \alpha', \alpha']$$

Now $a^{\alpha^{-1}} = a^{-1}bc^{-1}$, $b^{\alpha^{-1}} = b^{-1}ca^2$, $c^{\alpha^{-1}} = c^{-1}a^2b^2$, so $[a, \alpha^{-1}, \alpha^{-1}, \alpha^{-1}] = b^2c^2$. Let $\tilde\iota$ be the automorphism of A that takes each element of A to its inverse: Then $[a, \tilde\iota\alpha, \tilde\iota\alpha, \tilde\iota\alpha] = [c, \tilde\iota\alpha] = a^2c^2$, and $[a, \tilde\iota\alpha^{-1}, \tilde\iota\alpha^{-1}, \tilde\iota\alpha^{-1}] = a^2b^2c^2$. This implies that if $k \in \{1, 5\}$, then none of the elements $\alpha^k\tilde\iota$, α^{-k}, $\alpha^{-k}\tilde\iota$ is an $\mathrm{Aut}(A)$-conjugate of α; note that $[A, \alpha^4, \alpha^4] = 1$.

$N = N_{\mathrm{Aut}(A)}(\langle \alpha, \tilde\iota\rangle)$. Last paragraph's calculations yield $[N, \langle\alpha\rangle] \le \langle \alpha^4\rangle$. If $d \in \Phi(A)$, then there is an element of $C_N(\alpha)$ defined via $a \mapsto ad$, $b \mapsto b[d, \alpha]$, $c \mapsto [d, \alpha, \alpha]$. The automorphism thus determined being the unique element of $C_N(\langle \alpha, \Phi(A)\rangle)$ mapping a to ad, we find that $C_N(\langle\alpha, \Phi(A)\rangle) = \langle \alpha^4, \tilde\iota, \beta\rangle$ with $a^\beta = ac^2$, $[\beta, \langle b, c\rangle] = 1$. Certainly $[\beta, \tilde\iota] = 1$, so we may, setting $[\beta, \iota] = 1$, make β reemerge as an element of $Z(\mathrm{Aut}(G))$.

According to (3), we have $\alpha^5 = \alpha^\tau$ with $\tau \in \mathrm{Aut}(A)$ defined by $b^\tau = b^{-1}$, $\tau\,|_{\langle a, c, \Phi(A)\rangle} = \mathrm{id}$. For later use, we note that (3) also yields that

$$(4) \quad \begin{aligned} \alpha\tau &= \alpha^\psi, \psi \quad \text{defined via } a \mapsto a, b \mapsto b^{-1}, c \mapsto cb^2, \\ \alpha\beta &= \alpha^\psi, \psi \quad \text{given via } a \mapsto a, b \mapsto bc^2, c \mapsto c, \\ \alpha\beta\tau &= \alpha^\psi, \psi \quad \text{given via } a \mapsto a, b \mapsto b^{-1}c^2, c \mapsto c^{-1}. \end{aligned}$$

As $|Z(G)| = 2$, with each maximal subgroup U of G there is a unique automorphism ζ_U of G with $[U, \zeta_U] = 1$; the subgroup of $\mathrm{Aut}(G)$ consisting of the various ζ_U being isomorphic with (the dual of) $G/\Phi(G) = \langle \alpha\Phi(G), \iota\Phi(G), a\Phi(G)\rangle$, we see that it is generated by the maps $g \mapsto g^{b^2}$, $g \in G$, β and $\zeta := \zeta_{\langle \alpha, A\rangle}$. We have seen that $\langle \alpha^{\mathrm{Aut}(G)}\rangle \leq \langle \alpha\rangle A$, so $\langle \alpha\rangle A\, char\, G$ and $\zeta \in Z(\mathrm{Aut}(G))$.

Let A be embedded into the homocyclic group $B = \langle a, b, d\rangle$, with $d^2 = c$, and extend the action of $\langle \alpha, \iota\rangle$ to B by setting $d^\iota = d^{-1}$, $d^\alpha = da$. Using $(*)$, we see that $o(\alpha d) = 8$, because $[d, {}_7\alpha] = 1$. Next, $[\alpha a, \beta^d] = 1 = c^2[\alpha, [\beta, d]]$, while $[\iota c, \beta^d] = 1 = [\iota, [\beta, d]]$. Accordingly, $[\beta, \vartheta] \in b^2 C_{\mathrm{Aut}(G)}(\langle \alpha, \iota, A\rangle)$, whence $g^{[\beta, \vartheta]} = g^{b^2}$, $g \in G$. Furthermore, $[\alpha a, \tau^d] = \alpha^4[\alpha, [\tau, d]]$, $(\alpha a)^4 = \alpha^4 a^2$ and $[\iota c, \tau^d] = 1 = [\iota, [\tau, d]]$. Accordingly, $g^{[\tau, d]} = g^{\zeta c}$, $g \in G$.

If $\vartheta \in C_{\mathrm{Aut}(G)}(A)$, then $[\vartheta, G] \in C_G(A) = A$. One of the maps $g \mapsto g^{d\vartheta}$, $d \in A$, has $\alpha^{d\vartheta} \in \alpha\langle a\rangle$, we thus need only study the case $\alpha^\vartheta = \alpha a$. In this case $[\alpha, \vartheta^2] = a^2 = [\alpha, c]$. If $\iota^\vartheta = \iota e$ with $e \in A$, then $[\iota d, \alpha a] = 1 = a^2[d, \alpha]$; accordingly, $e \in \{c, c^{-1}\}$, and, as we are free to choose between ϑ and $\vartheta\zeta$, we may assume $[\iota, \vartheta] = c$. This shows that ϑ is the automorphism induced by the element d from the previous paragraph.

Now let $G \triangleleft K$ be a finite 2-group and let $C = C_K(G)$. Our analysis of $\mathrm{Aut}(G)$ implies that $K = GCU$ with $\Phi(U)(U \cap G) \leq \langle C, c, \Phi(A), z\rangle$, where $z^2 \in C$ and conjugation by z induces some element of $\langle \zeta\rangle$ on G. Furthermore, $[U, G] \leq \langle A, \alpha^4\rangle$ and $[\Phi(A), U] = 1$. Set $H = A\langle \alpha^4\rangle CU$; then $H \triangleleft G$, $K = H\langle \alpha, \iota\rangle$ and $H \cap \langle \alpha, \iota\rangle = \langle c^2, \alpha^4\rangle$. We would now like to produce $\sigma \in \mathrm{Aut}(K)$ with $G^\sigma \neq G$.

a) First suppose that $C \not\leq G$. Then there is $x \in C$ with $\langle x^2\rangle[x, K] \leq c^2$. For $h \in H$ s $\ell = \ell_h \in \{0, 1\}$ such that $[x, h] = c^{2\ell_h}$. Let σ be defined via $\alpha \mapsto \alpha x$, $\iota \mapsto \iota$ and $h \mapsto hb^{2\ell_h}$ whenever $h \in H$. Since $b^2 \in \Omega_1(Z(ACU))$, $\sigma|_H \in \mathrm{Aut}(H)$, while both definitions make σ act trivially on $H \cap \langle \alpha, \iota\rangle$. Clearly, $\langle \alpha, \iota\rangle \cong \langle \alpha, \iota\rangle^\sigma$, and if $h \in H$, then $[\alpha x, hb^{2\ell_h}] = [\alpha, h]c^{4\ell_h}$ while $[\iota, h^\sigma] = [\iota, h]$ anyway. If $\gamma \in \langle \alpha, \iota\rangle$, $h \in H$, then $[h, \gamma] \in C_H(x)$, so $[h, \gamma] = [h, \gamma]^\sigma$. Remark 1a) now shows $\sigma \in \mathrm{Aut}(K)$.

b) From now on we may assume that $K = \langle G, U\rangle$ where U is as in a). Suppose that there is $z \in U$ inducing ζ on G. As $\zeta \in Z(\mathrm{Aut}(G))$, there is $\ell = \ell_h \in \{0, 1\}$ such that $[z, h] = c^{2\ell_h}$ whenever $h \in U$. There is an isomorphism σ with $\alpha^\sigma = \alpha z$, $\iota^\sigma = \iota b^2$, $\sigma|_A = \mathrm{id}$, $z^\sigma = z$. Indeed, $(\iota b^2)^2 = \iota^2 = c^2$, ιb^2 inverts every element of A and $[\iota b^2, \alpha z] = [\iota z, b^2\alpha] = 1$, while $[d, \alpha z] = [d, \alpha]$, $d \in A$. So Remark 1a) applies. Finally, $[z, \langle A, \alpha\rangle^\sigma] = 1$ and $[z, \iota b^2] = c^2$. As $\sigma|_{U\cap G} = \mathrm{id}$,

σ may be extended to the whole of K by setting $h^\sigma = hb^{2\ell h}$; as $[U, b^2] = 1$, we find that $U^\sigma \cong U$ and $[h^\sigma, \iota^\sigma] = [h, \iota]$, $h \in U$. We have also made sure that $[\alpha^\sigma, h^\sigma] = [\alpha, h]$ for $h \in U$. Accordingly, $\sigma \in \mathrm{Aut}(K)$.

c) Suppose that $K = \langle G, u\rangle$ with $u^2 \in \langle c^2\rangle$ and conjugation by u inducing one amongst the set $\{\beta, \beta\zeta, \tau, \tau\beta, \tau\zeta, \tau\beta\zeta\}$ on G. Let $\alpha^\sigma = \alpha u$, and choose $\varphi = \sigma\,|_A$ according to the list given in (4); i.e. such that $(d^\varphi)^{\alpha u} = (d^\alpha)^\varphi$, $d \in A$. Please keep in mind that $[\zeta, A] = 1$, so (4) covers all possibilities. According to Remark 1a), σ induces an automorphism on $\langle A, \alpha\rangle$. Recall that $[\alpha, u] \in \langle \alpha^4, c^2\rangle \setminus \{1\}$. This easily implies $\alpha^4 = (\alpha u)^4$, so $[\alpha u, u] = [\alpha, u] = [\alpha, u]^\sigma$. Thus σ extends to an automorphism of $\langle \alpha, A, u\rangle$, setting $u^\sigma = u$.

Now suppose that $[\iota, u] = c^2$. Then $[\alpha u, \iota b^2] = 1$ and ιb^2 still inverts every element of A, whilst $\iota b^2 = c^2$. So we set $\iota^\sigma = \iota b^2$ in this case.

If $[\iota, u] = 1$, then $[\iota, \alpha u] = 1$; this time extend σ to the whole of K through setting $\iota^\sigma = \iota$.

d) Suppose that $K = \langle G, u, v\rangle$ u inducing one of β, $\tau\beta$, $\beta\zeta$, $\tau\beta\zeta$ and conjugation by v inducing τ on G. Then $[u, v] \in \langle c^2\rangle$. Define $\sigma\,|_{\langle G,\, u\rangle}$ as in c). Observe that $[\alpha u, v] = (\alpha u)^4[u, v]$, while one glance at (4) tells us that $[A, \sigma] \leq \Phi(A) \leq C_A(V)$. If $[u, v] = c^2$, then we let $v^\sigma = vb^2$, and we let $v^\sigma = v$ if $[v, u] = 1$. Remark 1a), applied to $Q = \langle G, u\rangle$, $S = \langle v\rangle$, says this produces an automorphism.

e) Next, suppose that there is $d \in K$ inducing ϑ or else $\vartheta\zeta$ on G. Recall that $g^{[\tau,\,\vartheta]} = g^{\zeta c}$, $g \in G$. By a), we have $K = \langle G, d, u\rangle$ where either $u = 1$, or u induces β or $\beta\zeta$ on G. We also know that we may take $d^2 = c$ and have noted that $o(\alpha d) = 8$, whence $\langle A, d, \alpha\rangle$ has an automorphism σ with $\alpha^\sigma = \alpha d$, $\sigma\,|_{\langle A,\, d\rangle} = \mathrm{id}$. Depending on whether d induces ϑ or $\vartheta\zeta$, either $[\iota b^{-1}, \alpha d] = 1$ or $[\iota b, \alpha d] = 1$. We may extend σ accordingly to obtain an automorphism of K and have now proved the lemma.

As stated before, I believe this example to be of minimal order. I also believe that giving a rigorous proof would be more tedious than rewarding. Here are some remarks: Let F have the desired property and let A be a maximal abelian normal subgroup of F. As $p = 2$, A cannot be cyclic — the reader is invited to verify this for himself. It turns out that A cannot be elementary abelian too, this is mainly because, if it was, either A itself would not be characteristic in F, or F could not contain elements acting on A as transvections. Letting $K = F * \langle c\rangle$, $1 \neq c^2 \in \Omega_1(G)$, we find $\Omega_1(F) < F$. If $\exp(A) \leq 4$ and $F = \Omega_1(F)\langle x\rangle$, with $o(x) = 4$ and x uniserial on A, while there is $\tau \in \mathrm{Aut}(F)\setminus\mathrm{Inn}(F)$ with $x^\tau = x^{-1}$,

$[\tau, \Omega_1(F)] \leq \Omega_1(A)$, then $F\, char\, F\langle\tau\rangle$. This leads to $|A| \geq 2^6$. The example was constructed with a view to avoid this situation. ∎

LEMMA 5: *Let p be an odd prime, and let P be a finite p-group; let $R \in Syl_p(\mathrm{Aut}(P))$. Assume that P satisfies the following conditions.*

$$(5)\qquad |Z(P)| = p, \text{ while } [R, P] \leq \Phi(P) \text{ and there is a complement } U \text{ of } \mathrm{Inn}(P) \text{ in } R \text{ of exponent } p.$$

Let $G = \langle\alpha, \iota\rangle$ be the group introduced in Lemma 2 and let $H = P \wr G$ be the wreath product of P with G with respect to the permutation representation $G \to S_{p^3}$ given by its action on the right cosets of $\langle\iota\rangle$. Then every property listed in (5) is enjoyed by H, and H is not characteristic in any finite p-group of which it is a proper subgroup.

Proof. We identify G with its image in S_{p^3}, labeling the elements of $\{1, \ldots, p^3\}$ in such a way that $\langle\iota\rangle$ is the stabiliser in G of $\{1\}$ and $\alpha = (1, 2, \ldots, p^3)$. Let $Q \cong P^{p^3}$ be the base group of H. For $i \in \{1, \ldots, p^3\}$ let $P_i = \{f : f \in Q,\, f(j) = 1 \text{ whenever } j \neq i\}$. Each element x in Q may be written as a p^3-tuple $x = (x_1, \ldots, x_{p^3})$ where each x_i is in P and $x^\gamma = (x_{1^{\gamma^{-1}}}, \ldots, x_{(p^3)^{\gamma^{-1}}})$ whenever $\gamma \in G$. If $\gamma \in G$, and $y = (y_1, \ldots, y_{p^3}) \in Q$, then $x^{y\gamma} = (z_1, \ldots, z_{p^3})$, where $z_i = x_{i^{\gamma^{-1}}}^{y_{i^{\gamma^{-1}}}}$. Thus $C_Q(y\gamma)$ is isomorphic with $C \times D$, where C is the direct product of the groups $C_{P_k}(y_k)$, k running over the set of fixed points of γ, and D is a direct product of ℓ copies of P, where ℓ is the number of orbits of $\langle\gamma\rangle$ of length greater than 1. As no element of $G \setminus \{1\}$ has more than p^2 fixed points, we find that, if $\gamma \in G \setminus \{1\}$, then $|C_Q(y\gamma)| \leq |P|^{2p^2-p}$, so $C_H(y\gamma) \leq p^5|P|^{2p^2-p}$. Since $p^{5+m(2p^2-p)} < p^{m(p^3-1)}$ for any natural number m, no element outside of Q can possibly be contained in an $\mathrm{Aut}(H)$-conjugate of P_1. Accordingly, $Q\, char\, H$. ∎

We now embark on a description of the Sylow p-subgroups of $\mathrm{Aut}(H)$. First of all, we utilise Remark 1d) to obtain $\mathrm{Aut}(H) = \mathrm{Inn}(H)N_{\mathrm{Aut}(H)}(\langle\alpha\rangle)$. If $\langle\iota'\rangle$ is a subgroup of H of order p^2 with $\alpha^{\iota'} = \alpha^{p+1}$, then $\iota' \in N_H(\langle\alpha\rangle) = GC_Q(\alpha)$ and, up to conjugation in G, we have $\iota' = \iota c$, $c \in C_Q(\alpha)$. Equivalently, $c = (y, \ldots, y)$ for some $y \in P$. We have worked out that $C_Q(\iota c) \cong C_P(y)^p \times P^{2p-2}$. Accordingly, ι' is not an $\mathrm{Aut}(H)$-conjugate of ι unless $y \in Z(P)$ or, equivalently, $c \in Z(Q)$. Let $C_{Z(Q)}(\alpha) = \langle z\rangle = Z(H)$; then there is an automorphism ζ of H with $\iota^\zeta = \iota z$, $\zeta\,|_{\langle\alpha,\, Q\rangle} = \mathrm{id}$. Now $z \in \Phi(Q) \leq \Phi(H)$.

Recall that the unique p-Sylow-subgroup of $\mathrm{Aut}(G)$ is equal to $\langle \mathrm{Inn}(G), \vartheta\rangle$ where $\alpha^\vartheta = \alpha\iota^p$ and $\iota^\vartheta = \iota$. We might as well suppose $\vartheta \in N_{S_{p^3}}(G)$, setting $1^{\gamma\vartheta} = 1^{\gamma^\vartheta}$, $\gamma \in G$.

Now Remark 1b) shows how to find $\tilde\vartheta$ in $N_{\mathrm{Aut}(H)}(G)$ with $\gamma^{\tilde\vartheta} = \gamma^\vartheta$, $\gamma \in G$. We identify $\tilde\vartheta$ and ϑ. Please note that $[\vartheta, \zeta] = \mathrm{id}$ and that $[P_1, \vartheta] = 1$ whence $[Q, \vartheta] \in [Q, G] \le \Phi(H)$, while we know that $[G, \vartheta] = \Phi(G)$.

Next, let τ be any p-element of $\mathrm{Aut}(H)$, $\tau \notin \mathrm{Inn}(H)$. We have seen that we may presume $\sigma\tilde\zeta \in N_{\mathrm{Aut}(H)}(G)$ for some power $\tilde\zeta$ of ζ, whence there are $\eta \in G$ and $\tilde\vartheta \in \langle\vartheta\rangle$ such that $\gamma^{\tau\tilde\zeta} = \gamma^{\eta\tilde\vartheta}$, whenever $\gamma \in G$.

Let $\Omega = \{P_1 Z(Q), \ldots, P_{p^3} Z(Q)\}$. According to Remark 1c), $\tau\tilde\zeta\tilde\vartheta^{-1}\eta^{-1}$ acts on Ω, and must induce some element of $C_{S_\Omega}(G)$. An abelian regular subgroup of S_Ω being its own centraliser in S_Ω, η may be multiplied by some element of $Z(G) = \langle\alpha^{p^2}\rangle$ such as to ensure that $\psi = \tau\tilde\zeta\tilde\vartheta^{-1}\eta^{-1}$ simultaneously centralises G and is trivial on Ω.

Let $C_{C_{\mathrm{Aut}(H)}(G)}(Q/Z(Q)) = V$; then $Z(P_1) = Z(Q) \cap P_1'$ is normalised by V, whence $[V, Z(Q)] = 1$, since $|Z(P)| = p$ by assumption. Accordingly, $[H, V, V] = 1$, and V is isomorphic to $\mathrm{Hom}(P/\Phi(P), C_{Z(Q)}(\iota))$, and, in particular, is elementary abelian. It is obvious that $[V, H] \le \Phi(H)$ and that $[V, \zeta] = 1$.

Back to dissecting τ : Let $Z(P_1) = \langle z_1\rangle$ and let $Y = \langle z_1^{\alpha^i} \mid 0 < i \le p^3 - 1\rangle$; then Y is a complement to $\langle z_1\rangle$ in $Z(Q)$ stabilised by ι. Let $\{q_1, \ldots, q_k\}$ be a minimal set of generators of P_1 and let $q_i^\psi = q_i' y_i$, $q_i' \in P_1$, $y_i \in Y$. The map $q_i \mapsto q_i y_i$, $1 \le i \le k$, is contained in $\mathrm{Aut}(P_1 Z(Q))$. For $1 \le i \le k$, we have $y_i \in C_Y(\iota)$, for ψ normalises $C_Q(\iota)$; accordingly, there is $\varphi \in V$ with $q_i^\varphi = q_i y_i$, $1 \le i \le k$, and $\psi\varphi^{-1}$ both centralises G and normalises P_1, thus is trivial on Ω. There is $\rho \in \mathrm{Aut}(P_1)$ such that $(x_1, \ldots, x_{p^3})^{\psi\varphi^{-1}} = (x_1^\rho, \ldots, x_{p^3}^\rho)$ whenever $(x_1, \ldots, x_{p^3}) \in Q$. Certainly $C_{\mathrm{Aut}(P_1)\wr\langle\alpha\rangle}(\alpha) \cong \mathrm{Aut}(P_1)$, in other words, every element ρ of $\mathrm{Aut}(P_1)$ gives rise to an element of $C_{\mathrm{Aut}(H)}(G)$ in this way. As $\langle\vartheta, \zeta, V\rangle \le C_{\mathrm{Aut}(H)}(H/\Phi(H)) \le O_p(\mathrm{Aut}(H))$, τ is a p-element if and only if ρ is. Accordingly, $[P, \rho] \le \Phi(P)$, so $[H, \tau] \le \Phi(H)$.

Now for the complement: Let $O_p(\mathrm{Aut}(P)) = \mathrm{Inn}(P)U$, $\exp(U) = p$, $U \cap \mathrm{Inn}(P) = 1$. Let $\widetilde U$ be the group of "diagonal" automorphisms $(x_1, \ldots, x_{p^3}) \mapsto (x_1^\rho, \ldots, x_{p^3}^\rho)$, $\rho \in U$, of Q. As a part of (5), we are assuming $[U, P] \le \Phi(P)$, so $[\widetilde U, Q] \le \Phi(Q)$. Now $[\Phi(Q), V] = 1 = [Z(Q), \widetilde U]$, whence $[Q, \widetilde U, V] = [Q, V, \widetilde U] = 1$. Thus $[\widetilde U, V] = 1$ by the Three-Subgroup-Lemma.

On Ω, $\langle \iota \rangle$ has p orbits of length one, and $p-1$ orbits of lengths p and p^2, respectively; thus ϑ stabilises each orbit of $\langle \iota \rangle$, hence $[C_{Z(Q)}(\iota), \vartheta] = 1$ — $Z(Q)$, after all, being the permutation module associated with the action of $\langle G, \vartheta \rangle$ on Ω. Accordingly, $[P_1, V\widetilde{U}, \vartheta] = 1 = [P_1, \vartheta, V\widetilde{U}]$. Certainly, $[V, \vartheta] \in C_{\mathrm{Aut}(H)}(\alpha)$, so $[V\widetilde{U}, \vartheta] = 1$.

If $v \in N_V(P_1)$, then v normalises each P_i and is the product of an inner automorphism induced by some element of $C_{Z_2(Q)}(G)$ and some element of $\widetilde{U}$. Let W be a complement in V to $N_V(P_1)$ and set $T = W\widetilde{U}\langle \vartheta, \zeta \rangle$. We have seen that $\langle W, \vartheta, \zeta \rangle \leq Z(T)$, in particular, $\exp(T) = p$. As $\vartheta \notin \mathrm{Inn}(G)$, $T \cap \mathrm{Inn}(H)$ is trivial on H/Q, and thus is contained in $W\widetilde{U}\langle \zeta \rangle$. Furthermore, $N_H(G) = C_Q(G)G$, G is faithful on Ω, while $W\widetilde{U}$ is trivial on Ω and $N_W(P_1) = 1$: An inner automorphism contained in $W\widetilde{U}$ would have to be contained in $\widetilde{U}$ and be induced by some element of $C_Q(G)$. As $C_Q(\alpha) = C_Q(G)$, no inner automorphism of H acts on G like ζ does. Thus $T \cap \mathrm{Inn}(H) = 1$.

Let $c \in C_{Z(Q)}(\langle \iota, \alpha^p \rangle)$, and let $d_j = [c, {}_j\alpha]$, $j \in \{0, \dots, p^3\}$. For every j, $d_j \in C_{Z(Q)}(\alpha^p)$, while $[d_{j+1}, \iota] = [d_j, \iota, \alpha]^{-\alpha}[\alpha^{-1}, \iota^{-1}, d_j]^{-\iota\alpha} = [d_j, \iota, \alpha]^{-\alpha}$. Via induction on j we thus obtain $[d_j, \iota] = 1$ for every j. Let $y_1 = [z_1, {}_{p^2-1}\alpha^p]$, $y_2 = [y_1, {}_{p^2-1}\iota]$, $y = [y_2, {}_{p-2}\alpha]$, $z = [y_2, {}_{p-1}\alpha]$. Then $\langle z \rangle = Z(H)$, $[y, \alpha] = z$, and, as we have just seen, $[y, \iota] = 1$. We are going to use the terms y_1, y_2, y, z throughout the remainder of this proof.

Let $H \lhd K$ with K a finite p-group, and $C = C_K(H)$. We have seen that $K = HCS$ with $\exp(S/S \cap C) = p$, and that $C_S(G)$ is trivial on Ω and of index at most p^2 in S. Furthermore, $SC \cap H = \langle z \rangle$. Again, we would like to find $\sigma \in \mathrm{Aut}(K)$ that does not stabilise H.

a) First assume that $C \not\leq H$; then there is $x \in C$ with $\langle [x, K], x^p \rangle \in \langle z \rangle$. Let $x^p = z^k$, $0 \leq k \leq p-1$. Let $\alpha^{\sigma_1} = \alpha x$, $\iota^{\sigma_1} = \iota y^{-k}$. Since $[\alpha x, \iota y^{-k}] = \alpha^p z^k = (\alpha x)^p$, σ_1 multiplicatively extends to an isomorphism (Remark 1a)). Next, let $\sigma_1 |_Q = \mathrm{id}$ and apply Remark 1a) to turn σ_1 into an automorphism of H— one just needs to point to the fact that $[Q, \langle x, y \rangle] = 1$.

If $u \in SC$, there is $\ell = \ell_u \in \{0, \dots, p-1\}$, with $[x, u] = z^{\ell_u}$; let $u^{\sigma_2} = uy^{\ell_u}$. Then $(SC)^{\sigma_2} \cong SC$, while both σ_1 and σ_2 centralise $H \cap SC = \langle z \rangle$. If $u \in SC$, and $g \in \langle Q, \iota \rangle$, then clearly $[u^{\sigma_2}, g] = [u, g]$; furthermore, $[uy^{\ell_u}, \alpha x] = [u, x][u, \alpha][y^{\ell_u}, \alpha] = [u, \alpha]$. If $u \in SC$ and $h \in H$, then $[u^{\sigma_1}, h^{\sigma_2}] = [u, h]$, and, as $[u, h] \in \langle \iota^p, \alpha^{p^2} Q \rangle \leq C_H(\sigma_1)$, we have $[u^{\sigma_2}, h^{\sigma_1}] = [u, h]^{\sigma_1}$. We may combine σ_1 and σ_2 into an element of $\mathrm{Aut}(K)$, once again using Remark 1a).

b) We are now entitled to assume that $C = \langle z\rangle$ and $K = HS$, with $\langle \mho_1(S), S\cap H\rangle \le \langle z\rangle$. Supposing that $C_S(G) \ne 1$ we find some element s such that $1 \ne s \in C_S(G)\cap Z(S/\langle z\rangle)$. Let $\ell \in \{0,\dots,p-1\}$ with $s^p = z^\ell$, and let $(\alpha^i)(\iota^j)^{\sigma_1} = (\alpha s)^i(\iota y^{-\ell})^j$, $i, j \in \mathbb{Z}$. Since $[\alpha s, \iota y^{-\ell}] = \alpha^p[\alpha, y^{-\ell}] = \alpha^p z^\ell = (\alpha s)^p$, Remark 1a) becomes available and σ_1 is an isomorphism. For $h = (x_1,\dots,x_{p^3}) \in Q$, let $h^{\sigma_1} = (x'_1,\dots,x'_{p^3})$ where $x'_i = (x_i)^{s^{(i-1)}}$, $1 \le i \le p^3$. Recalling that $P_i^s \le P_iZ(Q)$, for all i one readily sees that $\sigma_1|_Q \in \mathrm{Aut}(Q)$. Furthermore, $(x'_1,\dots,x'_{p^3})^{\alpha x} = ((x_{p^3})^{s^{p^3-1}}, x_1, x_2^s,\dots,(x_{p^3-1})^{s^{p^3-2}})^s = (x'_{p^3}, x'_1,\dots,x'_{p^3-1})$; in other words, $(h^{\alpha^i})^{\sigma_1} = h^{(\alpha s)^i}$, $h \in P_1$, $i \in \{0,\dots,p^3-1\}$. The action of $\iota y^{-\ell}$ on Q being completely determined by the facts that it centralises P_1 and raises αx to its $(p+1)$th power, the equation $(x'_1,\dots,x'_{p^3})^{\iota y^{-\ell}} = (x'_1, x'_{2^\iota-1},\dots,x'_{(p^3)^\iota-1})$ immediately follows. Again, (1) is satisfied, and σ_1 and σ_2 combine to $\sigma \in \mathrm{Aut}(\langle H\rangle)$.

We know that $S = C_S(G)\langle s, t\rangle$, where $\langle s, t\rangle\langle z\rangle/\langle z\rangle$ induces a subgroup of $\langle \vartheta, \zeta\rangle$ on H. Since $[z_1, \vartheta] = 1 = [\alpha^p, \vartheta]$, we have $[y_1, \vartheta] = 1 = [[y_1, {}_{p^2-1}\iota], \vartheta] = [y_2, \vartheta]$. Now $y = [y_2, {}_{p-2}\alpha]$, and we have seen that $[\iota, y_2] = 1$; accordingly $y^\vartheta = [y_2, {}_{p-2}(\alpha\iota^p)] = y$. Note that $[Q, \zeta] = 1$ and every element of $C_S(G)$ acts on $Z(P_1Z(Q))\cap\Phi(P_1Z(Q)) = \langle z_1\rangle$ and thus must be trivial on $Z(Q)$ anyway. If $u \in S$, then there is $\ell_u \in \{0,\dots,p-1\}$ such that $[s, u] = z^{\ell_u}$; let $u^{\sigma_2} = uy^{\ell_u}$, $u \in S$. Then $[\alpha s, u^{\sigma_2}] = [\alpha, u][s, u][\alpha, y^{\ell_u}] = [\alpha, u]$, thus $[\gamma, u^{\sigma_2}] = [\gamma, u] = [\gamma, u]^{\sigma_1}$ whenever $\gamma \in G$- recall that $[G, S] \le \langle \iota^p, \alpha^{p^2}\rangle \le C_H(\sigma_1)$. Let $q \in P_j$ for some $j \in \{1,\dots,p^3\}$. From $[s, u]$, $[u, \sigma_2] \in Z(Q)$, while $[q, u] \in P_jZ(Q)$ and $[Z(Q), s] = 1$ we derive that $(q^{\sigma_1})^{u^{\sigma_2}} = (q^{s^{j-1}})^u = q^{us^{j-1}} = (q^u)^{\sigma_1}$. Conjugation is a homomorphism, so $(q^{\sigma_1})^{u^{\sigma_2}} = (q^u)^{\sigma_1}$ whenever $q \in Q$, while, finally, for $q \in Q$, $\gamma \in G$, $((q\gamma)^{\sigma_1})^{u^{\sigma_2}} = ((q^{\sigma_1})^{u^{\sigma_2}}(\gamma)^{\sigma_1})^{u^{\sigma_2}} = (q^u)^{\sigma_1}(\gamma^u)^{\sigma_1} = (q\gamma)^{\sigma_1}$. Since $S\cap H \le \langle z\rangle$, Remark 1a) says that the map $hu \mapsto h^{\sigma_1}u^{\sigma_2}$, $h \in H$, $u \in U$, is contained in $\mathrm{Aut}(K)$.

We have shown that H is not characteristic in K unless $C_S(G) = 1$, which implies $|S| \le p^2$.

Suppose that $K = \langle H, s, t\rangle$, where s induces some element of $\zeta W\widetilde{U}$ on H and t induces some element of $\vartheta W\widetilde{U}$. Let $t^p = z^\ell$, and $[t, s] = z^{\ell_s}$ -naturally, $0 \le \ell, \ell_s \le p-1$. As in the final two paragraphs of the proof of Lemma 2, we may apply $(*)$ and infer that for $p > 3$, $(\alpha t)^p = \alpha^p t^p = [\alpha t, \iota y^{-\ell}]$ while if $p = 3$, then $(\alpha t)^3 = \alpha^3 t^3 \cdot \alpha^9 = [\alpha t, \iota^4 y^{-\ell}]$. We define σ_1 accordingly, setting $\alpha^{\sigma_1} \mapsto \alpha t$, $\iota^{\sigma_1} = \iota y^{-\ell}$, if $p > 3$, $\iota^{\sigma_1} = \iota^4 y^{-\ell}$ for $p = 3$. Applying Remark 1b), we see that σ_1 gives rise to an isomorphism. Extend σ_1 to $\langle G, s, t\rangle$ via $t \mapsto t$, $s \mapsto sy^{\ell_s}$. If $p = 3$, then $(\iota^4)^3 = \iota^3$, so $\alpha^{\sigma_1 t} = (\alpha t)^t = \alpha t(\iota^p) = \alpha^{\sigma_1}(\iota^{p\sigma_1})$, while $[\alpha t, sy^{\ell_s}] = [\alpha, y^{\ell_s}][t, s] = 1$. Clearly, $[\iota^{\sigma_1}, t] = 1$, while $[\iota^{\sigma_1}, s^{\sigma_1}] = [\iota, s] = z$. Applying Remark 1a) to the semidirect product $G\langle s, t\rangle$, we see that σ_1 becomes an isomorphism once we have extended it by demanding multiplicativeness.

The automorphism induced by t is a product $\vartheta\varphi$, $\varphi \in W\widetilde{U}$. Regard $\langle \alpha, \vartheta\rangle$ as a subgroup of S_{p^3}. Using $(*)$ like in the proof of Lemma 2, we find that $\alpha\vartheta = \beta$ is another p^3-cycle, hence there is $\xi \in S_{p^3}$ with $\alpha^\xi = \beta$. There $\sigma_2 \in \mathrm{Aut}(Q)$ given by $(x_1, \dots, x_{p^3}) \mapsto (x'_1, \dots, x'_{p^3})$, with $x'_i = x_{i\xi}^{\varphi^{(i-1)}}$, $1 \le i \le p^3$. Then $(x'_1, \dots, x'_{p^3})^{\alpha t} = (x'_{p^3}, x'_1, \dots, x'_{p^3-1})$. The action of the group $\langle t, \iota^{\sigma_1}\rangle$ on Q is fully determined by its centralising P_1 and its action on $\langle\alpha\rangle$. This implies that σ_1, σ_2, $\langle G, t\rangle$ (taking the role of S) and Q (in the role of Q) satisfy (1). Thus σ_1 and σ_2 combine to an isomorphism σ of $\langle H, t\rangle$; as $[Q, s] = 1$, and σ_1 is an isomorphism, setting $s^\sigma = s$ has σ extended into an element of $\mathrm{Aut}(K)$.

Finally, let $K = \langle H, s\rangle$, where the automorphism induced by s on H is in $\zeta^i\vartheta^j W\tilde{Q}$, $0 < i \le p-1$. We define σ_2 just as before, letting $\beta = \alpha\vartheta^j$, $\alpha^\xi = \beta$. Let $s^p = z^\ell$. The isomorphism σ_1 is defined via $\alpha \mapsto \alpha s$, $s \mapsto s$, $\iota \mapsto \iota y^{i-\ell}$, if $p > 3$, $\iota \mapsto \iota^4 y^{\ell-i}$, $p = 3$. If $p > 3$, then $[\alpha s, \iota y^{i-\ell}] = [s\alpha, \iota y^{i-\ell}] = \alpha^p z^i z^{\ell-i} = \alpha^p s^p = (\alpha s)^p$, while for $p = 3$ we get $[\alpha s, \iota^4 y^{i-\ell}] = [s\alpha, \iota^4 y^{i-\ell}] = \alpha^{12} z^\ell = (\alpha t)^3$. Thus σ_1 yields an isomorphism; Remark 1a) again delivers.

Lemma 6: *Let P be a nonabelian finite 2-group such that*

(6)

$$|Z(P)| = 2, \quad \text{every 2-automorphism of } P \text{ is trivial on } P/\Phi(P) \text{ and there is an elementary abelian complement } U \text{ to } \mathrm{Inn}(P) \text{ in a Sylow-2-subgroup of } \mathrm{Aut}(P).$$

Let H be the wreath product $H = P \wr D$, $\langle \alpha, \iota\rangle = D \cong D_8$, $o(\alpha) = 4$, $o(\iota) = 2$, $\alpha^\iota = \alpha^{-1}$ with respect to the action of D on the right cosets of $\langle\iota\rangle$ in D. Then H inherits each of the properties listed in (6), *while there is no finite 2-group K properly containing H as a characteristic subgroup.*

Proof. First of all, $|Z(H)| = 2$, and $Z(Q)$ is the permutation module over $GF(2)$ with respect to the prescribed embedding $D \to S_4$. Let $Q \cong P^4$ be the base group of H, and write the elements of Q as quadruples $(x_1, \dots, x_4)$, $x_1, \dots, x_4 \in P$, with $(x_1, \dots, x_4)^\delta = (x_{1^{\delta^{-1}}}, \dots, x_{4^{\delta^{-1}}})$, $\delta \in D$. For $1 \le i \le 4$, let P_i be the group of quadruples of elements of P with all entries equal to

1 apart from possibly the ith one. If $g \in P$ and $q = (g, 1, 1, 1) \in P_1$, then $C_Q(q) \cong C_P(g) \times P^3$, while $C_H(q) = C_Q(q)\langle\iota\rangle$. For $x \in H \setminus Q$, $|C_Q(x)| \le |P|^3$, and $|C_Q(x)| = |P|^3$ if and only if $x = \delta y$, where δ is a noncentral involution in D and $y \in Z(Q)$. Accordingly, $C_H(x) \le \langle\alpha^2, x\rangle C_Q(x)$, and x and q cannot be $\mathrm{Aut}(H)$-conjugates. Thus $Q\, char\, H$.

With regards to methods as well as to results, the analysis of 2-automorphisms of H proceeds much as its counterpart in Lemma 5, thus is presented more succinctly. As before, Remark 1d) yields $\mathrm{Aut}(H) = \mathrm{Inn}(H)N_{\mathrm{Aut(H)}}(\langle\alpha\rangle)$. In the present circumstances, this means $\mathrm{Aut}(H) = \mathrm{Inn}(H)C_{\mathrm{Aut}(H)}(\alpha)$. If $\varphi \in C_{\mathrm{Aut}(H)}(\alpha)$, then $[\varphi, \iota] \in N_Q(\langle\alpha\rangle) = C_Q(\langle\alpha, \iota\rangle)$; taken together with $|C_Q(\iota^\varphi)| = |C_Q(\iota)|$ this implies $[\iota, \varphi] \in Z(H)$. Let $Z(H) = \langle z\rangle$. There is $\zeta \in \mathrm{Aut}(H)$ defined by $\iota^\zeta = \iota z$, $\zeta|_{\langle\alpha, Q\rangle} = \mathrm{id}$; we know at this point that $\mathrm{Aut}(H) = \mathrm{Inn}(H)C_{\mathrm{Aut}(H)}(D)\langle\zeta\rangle$. Note that $\zeta \in Z(\mathrm{Aut}(H))$ and $[\zeta, H] \le \Phi(H)$.

According to Remark 1c), $\mathrm{Aut}(H)$ acts on the set $\Omega = \{P_iZ(Q) : 1 \le i \le 4\}$, and if $\psi \in C_{\mathrm{Aut}(H)}(G)$, then we may take ψ to be trivial on Ω, we could multiply by the inner automorphism induced by α^2 otherwise. Let

$$V = C_{C_{Aut(H)}(D)}(Q/Z(Q));$$

as $Z(P_1) = Z(P_1Z(Q)) \cap (P_1Z(Q))'$ $[V, Z(Q)] = 1$, whence V is elementary abelian and isomorphic to $\mathrm{Hom}(P/\Phi(P), C_{Z(Q)}(\iota))$. Observe that $[V, \zeta] = 1$ and $[V, H] \le \Phi(H)$.

Let $Z(P_i) = \langle z_i\rangle$, $1 \le i \le 4$, and let $Y = \langle z_2, z_3, z_4\rangle$. If $\psi \in C_{\mathrm{Aut}(H)}(G)$ and ψ is trivial on Ω, then $[\psi, P_1] \le P_1C_Y(\iota)$, there is $v \in V$ such that $\psi v N_{C_{\mathrm{Aut}(H)}(G)}(P_1)$. As $\langle V, \zeta\rangle \le O_2(\mathrm{Aut}(H))$, each 2-automorphism of H is a product $\varphi\eta$, where $\eta \in \langle \mathrm{Inn}(H), \zeta, V\rangle$ and φ is a 2-element of $N := N_{C_{\mathrm{Aut}(H)}(G)}(P_1)$. If $\rho \in U$, then let $\tilde\rho$ be the automorphism of Q defined by $(x_1, x_2, x_3, x_4) \mapsto (x_1^\rho, \dots, x_4^\rho)$; the map $\rho \mapsto \tilde\rho$, $\rho \in \mathrm{Aut}(P_1)$, is an isomorphism; as $C_N(P_1) = 1$, $N = \{\tilde\rho : \rho \in \mathrm{Aut}(P)\}$. Let $\widetilde{U} = \{\tilde\rho : \rho \in U\}$. Then $[\widetilde{U}, Z(Q)] = 1 = [V, \Phi(Q)]$, condition (6) says $[\widetilde{U}, H] \in \Phi(H)$, so $1 = [Q, V, \widetilde{U}][Q, \widetilde{U}, V]$ so $[\widetilde{U}, V] = 1$.

Through replacing V by a complement W of $N_V(P_1)$ in V, we thus obtain an elementary abelian supplement $L = W\widetilde{U}\langle\zeta\rangle$ of $\mathrm{Inn}(H)$ in a Sylow 2-subgroup of $\mathrm{Aut}(H)$. Let $\psi \in L \cap \mathrm{Inn}(H)$ be conjugation by the element h. Then $C_H(\alpha) \cap N_H(\langle\iota, z\rangle) = \langle\alpha^2, C_Q(\alpha)\rangle$, so $[\psi, \iota] = 1$ and $\psi \in W\widetilde{U}$. This in turn implies $h \in C_Q(\alpha)$, so $\psi \in \widetilde{U}$; all in all, $L \cap \mathrm{Inn}(H) \cong U \cap \mathrm{Inn}(P) = 1$.

Now let $H \triangleleft K$ be a finite 2-group and let $C = C_K(H)$. If $C \not\leq H$, then there is $x \in C$ such that $\langle x^2 \rangle [x, K] \leq Z(H)$. We know that $K = HM$, where $C \subseteq M$ and M/C induces a subgroup of L on H. If $m \in M$, then there is $\ell_x \in \{0, 1\}$ such that $[x, m] = z^{\ell_m}$. Let $\langle z_1 \rangle = Z(P_1)$ and let $y = z_1 z_1^{\alpha^2}$, Then $[y, \alpha] = z$, while $[\eta, \langle \iota\, \alpha^2, Q \rangle] = 1 = [y, L]$. Define σ_1 via $\alpha \mapsto \alpha x$, $\iota \mapsto \iota y^k$, where $x^2 = z^k$, $k \in \{0, 1\}$, and $\sigma_1 |_Q = \mathrm{id}$. Since $[\alpha y, \iota y^k] = \alpha^2 z^k$, imposing multiplicativeness makes σ_1 an isomorphism. Let $\sigma_2 |_M$ be defined via $m \mapsto m y^{\ell_x}$, $m \in M$. As $[y, M] = 1$, σ_2 is an isomorphism and from what we have learned about $L \cap \mathrm{Inn}(H)$ in particular, that it consists of inner automorphisms induced by elements of $C_Q(D)$ — we know that σ_1 and σ_2 are trivial on $H \cap M$ as well as $[H, M]$. Thus Remark 1a) yields that $\sigma : K \to K$, defined via $(xh)^\sigma = x^{\sigma_1} h^{\sigma_2}$, $h \in H$, $x \in M$, is in $\mathrm{Aut}(H)$.

We may now assume that $K = HM$, with $\Phi(M)(H \cap M) \leq \langle z \rangle$. If there is $s \in M$ with s inducing some element of $\widetilde{U}W$ on H, then we define σ_1 via $\alpha \mapsto \alpha s$, $\iota \mapsto \iota y^k$ with $s^2 = z^k$. Now $[\alpha s, \iota y^k] = \alpha^2 z^k = (\alpha s)^2$, so σ_1 gives rise to an isomorphism. We extend σ_1 via $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^s, x_3, x_4^s)$, $(x_1, x_2, x_3, x_4) \in Q$; as $(x_1, x_2^s, x_3, x_4^s)^{\alpha s} = (x_4^s, x_1, x_2^s, x_3)^s = (x_4, x_1^s, x_2, x_3^s)$, Remark 1a) says σ_1 is an isomorphism. If $m \in M$, then there is $\ell_m \in \{0, 1\}$ such that $[s, m] = z^{\ell_m}$ and we let $m_2^\sigma = m y^{\ell_m}$, $m \in M$. Again, $\sigma_2 |_H$ is an isomorphism, while, for $m \in M$, $[\alpha s, m y^{\ell_m}] = [s, m][\alpha, y^{\ell_m}] = 1 = [\alpha, m]$ and $[\iota, m y^{\ell_m}] = [\iota, m]$; if $i \in \{1, 2, 3, 4\}$ and $q \in Q_i$, then $[q^{s^{i-1}}, m y^{\ell_m}] = q^{m s^{i-1}}$. As $[H, M] \leq \Phi(Q) \leq C_Q(\sigma_1)$, Remark 1a) again comes into play and proves σ to be contained in $\mathrm{Aut}(K)$.

The only possibility left for us to consider is $K = H\langle s \rangle$, where s induces $\zeta \tilde{\rho} v$ on H, where $\rho \in U$, $v \in V$. On $\langle \alpha, Q, s \rangle$ define σ exactly as before (in particular, $s^\sigma = s$). Let $s^2 = z^k$, $k \in \{0, 1\}$ and let $\iota^\sigma = \iota y^{k+1}$. Since $[\alpha s, \iota y^{k+1}] = \alpha^2 z^{k+1} z = \alpha^2 z^k$, Remark 1a) may be brought forward once more to show $\sigma \in \mathrm{Aut}(K)$. ∎

Proof of the Theorem. As is well-known (see [4], 15.3) the Sylow-p-subgroups of S_{p^n} are isomorphic with the n-fold wreath product $\mathbb{Z}/p\mathbb{Z} \wr \ldots \wr \mathbb{Z}/p\mathbb{Z}$. If p is odd, let G be the group from Lemma 2. Using this lemma and letting Lemma 5 provide the inductive step, we obtain that, for $n \in \mathbb{N}$, the n-fold wreathed product $G_n = G \wr \ldots \wr G$, (n times) with G embedded into S_{p^3} as in 3, is not characteristic in any finite p-group properly containing it. If G_n contains a subgroup isomorphic to the n-fold wreath product $\mathbb{Z}/p\mathbb{Z} \wr \ldots \wr \mathbb{Z}/p\mathbb{Z}$, then, as

$P \wr G$ has a subgroup isomorphic to the regular wreath product $P \wr \langle \alpha^{p^2} \rangle$, $P \wr G$ contains an isomorphic copy of a Sylow-p-subgroup of $S_{p^{n+1}}$.

Now for $p = 2$: In the semidihedral group $P = \langle \eta, \delta | \eta^8 = 1 = \delta^2, \eta^\delta = \eta^3 \rangle \cong SD_{16}$, we have $\Phi(P) = \langle \eta^2 \rangle$, $\eta\Phi(P)$ is comprised entirely of elements of order 8, $\eta\delta\Phi(P)$ consists of elements of order 4, while every element of $\delta\Phi(P)$ is an involution. Thus $\mathrm{Aut}(P)$ is trivial on $P/\Phi(P)$, and, moreover, acts on $\langle \delta \rangle^P$. Thus if $\zeta \in \mathrm{Aut}(P) \setminus \mathrm{Inn}(P)$, ζ may be taken to centralise δ and normalise $\langle \eta \rangle$. Multiplying by δ, if necessary, we find $\eta^\zeta = \eta^5$, $\delta^\zeta = \delta$. Accordingly, the group SD_{16} is fit to play the role of P in Lemma 6; this lemma then inductively yields that the n-fold wreath products $SD_{16} \wr D_8 \wr \ldots \wr D_8$ (with respect to the embedding $D_8 \to S_4$) are never characteristic in finite 2-groups properly containing them. Arguing as for odd p, we see that an n-fold wreathed product of this kind has a subgroup isomorphic with the n-fold wreath product $\mathbb{Z}/2\mathbb{Z} \wr \ldots \wr \mathbb{Z}/2\mathbb{Z}$. This proves the theorem. ∎

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